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LETTER TO THE EDITOR

Largest eigenvalue distribution in the double scaling limit of matrix models: a Coulomb fluid approach

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Abstract. Using thermodynamic arguments we find that the probability that there are no eigenvalues in the interval $(-s, \infty)$ in the double scaling limit of Hermitian matrix models is $O(\exp(-s^{2\gamma+2}))$ as $s \to +\infty$. Here $\gamma = m - 1/2$, $m = 1, 2, \ldots$ determines the mth multicritical point of the level density: $\sigma(x) \propto b[1-(x/b)^2]^\gamma$, $x \in (-b, b)$ and $b^2 \propto N$. Furthermore, the size of the transition zone, where the eigenvalue density becomes vanishingly small at the tail of the spectrum, is $\approx N^{(\gamma-1)/2(\gamma+1)}$ in agreement with earlier work based upon the string equation.

A basic quantity in Hermitian matrix models is the probability, $E_2(0; J)$, that a set J contains no eigenvalues. For $N \times N$ Hermitian matrix models with unitary symmetry we have the well known expression

$$E_2(0; J) = \frac{\int_{\bar{J}} \exp\left[-\sum_a V(x_a)\right] \mathrm{d}\mu(x)}{\int_{J\cup\bar{J}} \exp\left[-\sum_a V(x_a)\right] \mathrm{d}\mu(x)} =: \frac{Z[\bar{J}]}{Z[J \cup \bar{J}]} =: \exp[-(F[\bar{J}] - F[J \cup \bar{J}])] \tag{1}$$

with

$$\mathrm{d}\mu(x) = \prod_{1\leqslant a<b\leqslant N} |x_a - x_b|^2 \prod_{1\leqslant a\leqslant N} \mathrm{d}x_a$$

where $\bar{J}$ is the complement of J and $V(x)$ is the 'confining' potential [1]. As indicated in (1), minus the logarithm of this probability has the physical interpretation, in terms of Dyson's Coulomb fluid [2–5], as the change in free energy

$$\Delta F = F[\bar{J}] - F[J \cup \bar{J}] \tag{2}$$

that is, the free energy of the N charges confined to region $\bar{J}$ minus the free energy of N charges in the natural support $J \cup \bar{J}$ of $w(x) := \mathrm{e}^{-V(x)}$.

In this letter we shall mainly consider the case $J = (-s, \infty)$, $s > 0$, and write $E_2(s)$ for $E_2(0; (-s, \infty))$. We shall use the *continuum approximation* of Dyson [2] which treats the N eigenvalues in the large N limit as a continuous fluid described by a continuous charge density σ with the free energy expressed in terms of σ. This approximation has previously been applied to the unitary Laguerre ensemble (ULE) where $w(x) = x^\alpha \mathrm{e}^{-x}$, $x \in (0, \infty)$ and

$\alpha > -1$ [5]†. Here we examine matrix models with‡

$$V(x) = \sum_{k=0}^{p} \frac{g_{2k+2}}{(k+1)b^{2k}} x^{2k+2} \tag{3}$$

with $g_2 = 1$. In principle, we should not have to make the continuum approximation since it is known that $E_2(0; J)$ is expressible in terms of solutions to a completely integrable system of partial differential equations [6]. However, the analysis of these equations for V of the form given by (3) is quite difficult. (Of course, the Gaussian case is not included in this remark.) It is hoped that the approximate expressions derived here, which we believe are asymptotic as $s \to \infty$, will aid in the analyses of these equations.

To begin, consider the Gaussian unitary ensemble (GUE) with $g_{2k+2} = 0$ for $k \geqslant 1$. For the scaled GUE with $J = (-t, t)$ it is known that $E_2(0; (-t, t))$ is a τ-function of a particular fifth Painlevé transcendent [7]. Starting with this representation, an asymptotic expansion for $E_2(0; (-t, t))$ as $t \to \infty$ can be derived, though the first such asymptotic expansion was achieved by Dyson using methods of inverse scattering [8]. (Actually, there is still an undetermined constant from either the inverse scattering analysis or the Painlevé analysis, see e.g. [9].) The leading term, $-\ln E_2(0; (-t, t)) \sim \pi^2 t^2/2$, was first obtained from the fluid approximation [2]. Indeed, the t^2 term of the asymptotic expansion can be given a simple physical interpretation: it is proportional to the square of the number of eigenvalues excluded in the (scaled) interval $(-t, t)$, since in the bulk scaling limit of the GUE the eigenvalue density is a constant $\sim \sqrt{2N}/\pi$. This suggests that a natural variable is one which gives uniform density in the excluded interval. We can always achieve this by a simple change of variables since the problem is one-dimensional. By introducing a new variable ξ and a corresponding $\rho(\xi)$ via the relation

$$\rho(\xi)\,\mathrm{d}\xi := 1\mathrm{d}\xi = \sigma(x)\,\mathrm{d}x \tag{4}$$

the density in the ξ 'scale' is made unity. Therefore, $-\ln E_2(0; J)$ is asymptotic to

$$\left[\int_{\xi_1}^{\xi_2} \mathrm{d}\xi\right]^2 = \left[\int_{x_1}^{x_2} \sigma(x)\,\mathrm{d}x\right]^2 \qquad J = (x_1, x_2).$$

We conclude from the above arguments that for a large interval

$$-\ln E_2(0; J) \sim N^2(l) \tag{5}$$

where

$$N(l) = \text{number of eigenvalues excluded in an interval of length } l. \tag{6}$$

We mention that a screening theory of the continuum Coulomb fluid gives a physical justification of these arguments [10], though we do not know of a proof of the general validity of this relationship.

† It is known from the theory of liquids (by an application of the Boltzmann principle) that the probability, $P_d(R)$, of finding a bubble of radius R in the bulk of a fluid (in d dimensions) at equilibrium with temperature $1/\beta$ is

$$P_d(R) \sim \exp\left[-\beta E_V R^d - \beta E_{\partial V} R^{d-1}\right] \qquad R \gg \text{coherence length}$$

where E_V is the energy/volume for creating a bubble and $E_{\partial V}$ is the surface energy. If we specialize this formula to $d = 1$ then

$$P_1(R) \sim \exp[-\text{constant } R]$$

in contradiction with the known result [2]. This is due to the fact that the Coulomb fluid has long ranged interactions.

‡ The reason for our choice of notation for the coefficients of V will become clear below.

To further test the validity of (5) and (6) we consider the edge scaling limit of the GUE where exact results are known [11]. Accordingly, we simply compute the number of eigenvalues excluded from an interval of length $l(= b - a)$ from the soft edge $b = \sqrt{2N}$:

$$N(l) = \int_a^b \mathrm{d}x\, \frac{1}{\pi}\sqrt{b^2 - x^2} \propto \sqrt{2b}\int_a^b \mathrm{d}x\,\sqrt{b-x} \propto [2^{1/2}N^{1/6}l]^{3/2} =: s^{3/2}.$$

Observe that $N^2(l) \sim s^3$ not only supplies the correct exponent in the scaled variable s $(= 2^{1/2}N^{1/6}l)$ in $-\ln E_2(s)$, we also have the correct density at the soft edge $\sigma_N(\sqrt{2N}) = 2^{1/2}N^{1/6}$, which agrees with known exact results [1, 11]. This result predicts the shrinking of the size of the transition zone ($\sim N^{-1/6}$) as $N \to \infty$—a reasonable behaviour from the Coulomb fluid point of view since the GUE potential x^2 is strongly confining. The same approximation has been applied to the origin scaling limit of the ULE [5] and the result agrees with the first term of the exact asymptotic expansion [12].

These two confirmations of the validity of (5) and (6) give us confidence to apply the method to the matrix models with V given by (3). (These are the cases of interest in the matrix models of two-dimensional quantum gravity [13, 14].) The charge density σ satisfies an integral equation [2, 3] derived from the following minimum principle:

$$\min_{\sigma} F[\sigma]$$
$$F[\sigma] = \int_J \mathrm{d}x\, V(x)\sigma(x) - \int_J \mathrm{d}x \int_J \mathrm{d}y\, \sigma(x)\ln|x-y|\sigma(y) \tag{7}$$

subject to the constraint $\int_J \mathrm{d}x\,\sigma(x) = N$, which is

$$V(x) - 2\int_{-b}^{b} \mathrm{d}y\, \ln|x-y|\sigma(y) = \text{constant} = \text{chemical potential} \qquad x \in (-b, b). \tag{8}$$

Since V is even so is σ. Making use of this symmetry (8) becomes

$$V(x) - 2\int_0^b \mathrm{d}y\, \ln|x^2 - y^2|\sigma(y) = \text{constant}. \tag{9}$$

With the change of variables $x^2 = u$, $y^2 = v$ and $r(u) = \sigma(\sqrt{u})/(2\sqrt{u})$, (9) becomes

$$V(\sqrt{u}) - 2\int_0^{b^2} \mathrm{d}v\, r(v)\ln|u-v| = \text{constant} \qquad u \in (0, b^2). \tag{10}$$

This is converted into a singular integral equation by differentiating with respect to u:

$$\frac{\mathrm{d}V(\sqrt{u})}{\mathrm{d}u} - 2\mathrm{P}\int_0^{b^2} \mathrm{d}v\, \frac{r(v)}{u-v} = 0 \qquad u \in (0, b^2). \tag{11}$$

Here b, which determines the upper and lower band edges, is fixed by the normalization condition $\int_{-b}^{b} \sigma(x)\,\mathrm{d}x = N$.

Following [15] the solution is†

$$\begin{aligned} r(u) &= \frac{1}{2\pi^2}\sqrt{\frac{b^2-u}{u}}\,\mathrm{P}\int_0^{b^2} \frac{\mathrm{d}v}{v-u}\sqrt{\frac{v}{b^2-v}}\frac{\mathrm{d}V(\sqrt{u})}{\mathrm{d}u} \qquad u \in (0, b^2) \\ &= \sqrt{\frac{b^2-u}{u}}\sum_{k=0}^{p} t_k\, {}_2F_1\left(-k, 1, \tfrac{3}{2}, 1 - \frac{u}{b^2}\right) \end{aligned} \tag{12}$$

† Constant$/\sqrt{u(b^2-u)}$ solves the homogeneous part of (11). However, based on the variational principle, including this solution would increase the free energy. ${}_2F_1(-k, 1, 3/2, z) = \sum_{n=0}^{k}[(-k)_n/(3/2)_n]z^n$ is a polynomial of degree k in z.

where the integral can be found in [16] and

$$t_k := -\frac{1}{2\pi^2} B(-\tfrac{1}{2}, k + \tfrac{3}{2}) g_{2k+2}.$$

Returning to σ, it can be shown that

$$\sigma(x) = b\sqrt{1 - \left(\frac{x}{b}\right)^2}\,\Pi_p\left[\left(\frac{x}{b}\right)^2\right] \tag{13}$$

where $\Pi_p(z)$ is a polynomial of degree p in z with coefficients depending on the linear combinations of the coupling constants g_k. The edge parameter b is determined from the normalization condition and reads $b^2 = CN$ where

$$C = \frac{1}{\int_{-1}^{+1} \mathrm{d}t\, \sqrt{1-t^2}\,\Pi_p(t^2)}$$

is independent of N.

Taking the special case $p = 1$ (now $g_4 = g$), we have

$$\sigma(x) = \frac{b}{\pi}\sqrt{1 - \left(\frac{x}{b}\right)^2}\left[1 + \frac{g}{2} + g\left(\frac{x}{b}\right)^2\right].$$

By tuning g to g_c, such that $-g_c = 1 + g_c/2$, we have

$$\sigma(x) = \text{constant } b\left[1 - \left(\frac{x}{b}\right)^2\right]^{3/2}$$

producing a qualitative deviation in the density at the edges ($\pm b$) of the spectrum from Wigner's semi-circle distribution [17, 14]. A calculation now gives

$$N(l) \propto \int_a^b \mathrm{d}x\, b(1 - x/b)^{3/2}(1 + x/b)^{3/2} \approx \frac{b}{b^{3/2}} \int_a^b (b - x)^{3/2} \sim (l/N^{1/10})^{5/2} =: s^{5/2} \tag{14}$$

and thus $-\ln E_2(l) \sim s^5$. Observe that due to this tuning, the length of the transition zone ($\sim N^{1/10}$) is now an increasing function of N. It is clear that the tuning procedure can be generalized to $p > 1$ [13]. By simultaneously adjusting the coupling constants g_4, g_6 etc, to their respective critical values we can have

$$\sigma(x) = \text{constant } b\left[1 - \left(\frac{x}{b}\right)^2\right]^{\gamma} \tag{15}$$

where $\gamma = p + \frac{1}{2}$†. Computing $N(l)$ we find

$$N(l) \propto \int_a^b \mathrm{d}x\, b\left(1 - \frac{x}{b}\right)^{\gamma}\left(1 + \frac{x}{b}\right)^{\gamma} \propto \left(\frac{l}{N^{(\gamma-1)/2(\gamma+1)}}\right)^{\gamma+1} =: s^{\gamma+1}. \tag{16}$$

Therefore,

$$\ln E_2(s) \approx -s^{2\gamma+2} \qquad (s \to \infty). \tag{17}$$

The non-perturbative soft edge density is determined as

$$\sigma_N(\sqrt{N}) \approx N^{(1-\gamma)/2(1+\gamma)} \qquad N \to \infty. \tag{18}$$

The corresponding size of the transition zone is $\approx N^{\mu}$, where

$$\mu = (\gamma - 1)/2(\gamma + 1)$$

† In quantum gravity literature $\gamma = m - 1/2$, $m = 1, 2, \ldots$.

a result previously obtained from the string equation [17, 18]. Note that our x variable is related to Bowick and Brézin's λ as $x = \sqrt{N}\lambda$ [17]. Supplying the appropriate $\sqrt{N}$ factor we obtain from (18) Bowick and Brézin's result $N^{-2/(2m+1)}$.

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